MATH1520 University Mathematics for Applications **Fall 2021**

Chapter 6: Application of Derivatives I

Learning Objectives:
(1) Apply L'Hôpital's rule to find limits of indeterminate forms.
(2) Discuss increasing and decreasing functions.
(3) Define critical points and relative/absolute extrema of real functions of 1 variable.
(4) Use the first derivative test to study relative/absolute extrema of functions.

6.1 Limits of indeterminate forms and L'Hôpital's rule

Recall the Remark in the end of Section 2.4 regarding exceptional cases of limits, which can not be computed using the algebraic rules of limits in Proposition 2, but the limits might still exist. Limits of this type are said to be of indeterminate forms.

6.1.1 Limits of indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$

Consider $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$,

1. if $\lim\limits_{x\to a} f(x) = A$, $\lim\limits_{x\to b} g(x) = B \neq 0$, $A, B \in \mathbb{R}$, then by the quotient rule,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{A}{B}.$$

2. if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0 \quad (\pm\infty)$, then the quotient rule is not applicable. Limits of this type are said to be of indeterminate form type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$

   For example,

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1}, \quad \left(\text{type } \frac{0}{0}\right)$$

$$\lim_{x\to +\infty} \frac{x+1}{2x+3}, \quad \lim_{x\to +\infty} \frac{-x+1}{2x^3}, \quad \left(\text{type } \frac{\infty}{\infty}\right).$$

Theorem 6.1.1 (L'Hôpital's rule for limits of types $\frac{0}{0}, \frac{\infty}{\infty}$).

Let $f(x)$, $g(x)$ be differentiable and suppose that $g'(x) \neq 0$ near the point a.

If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \textit{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Remark. (a) An intuitive explanation: When $f(a) \approx 0 \approx g(a)$,

$$\frac{f(x)}{g(x)} \approx \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

(b) The statement of the theorem still holds if "$x \to a$" is replaced by "$x \to \pm\infty$" or "$x \to a^{\pm}$". It also holds if $\lim_{x\to a} f(x) = \pm\infty$ $\lim\limits_{x\to a} g(x) = \mp\infty$. (Use $\lim_{x\to a} \frac{f(x)}{g(x)} = -\lim_{x\to a} \frac{-f(x)}{g(x)}$ and apply the theorem to $\lim_{x\to a} \frac{-f(x)}{g(x)}$.)

Example 6.1.1. Limits of type $\frac{0}{0}$

1.

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1} \qquad \text{(check condition 1: } \tfrac{0}{0}\text{)}$$
$$= \lim_{x\to 1} \frac{2x}{3x^2} \qquad \text{(check condition 2: this limit is } \tfrac{2}{3}\text{)}$$
$$= \frac{2}{3}.$$

$= \lim_{x \to 1} \frac{2}{3x}$ quotient

Remark. Alternatively, use the "canceling common factors" trick in the previous chapters.

2.

$$\lim_{x\to 1} \frac{e^x - e}{\sqrt{x}-1} \qquad \text{(the limit is of type } \tfrac{0}{0}\text{)}$$
$$= \lim_{x\to 1} \frac{e^x}{\frac{1}{2}x^{-\frac{1}{2}}}$$
$$= 2e.$$

quotient rule. $= \dfrac{\lim_{x\to 1} e^x}{\lim_{x\to 1} \frac{1}{2}\frac{1}{\sqrt{x}}} = \dfrac{e}{\frac{1}{2}}$

3.

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x^2} \qquad (\text{type } \frac{0}{0})$$
$$=\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{2x}$$
$$=+\infty.$$

$\frac{d}{dx}\ln(1+x)$ $\quad u=1+x$

$=\frac{d\ln u}{dx}$

$=\frac{d\ln u}{du}\frac{du}{dx}=\frac{1}{u}\cdot 1$

$=\frac{1}{1+x}$

Example 6.1.2. Limits of type $\frac{\infty}{\infty}$

1.

$$\lim_{x\to+\infty}\frac{-x+1}{2x+3} \qquad (\text{type } \frac{\infty}{\infty})$$
$$=\lim_{x\to+\infty}\frac{-1}{2}$$
$$=-\frac{1}{2}.$$

Remark. The same result can be obtained by dividing both the numerator and the denominator by x.

2.

$$\lim_{x\to+\infty}\frac{\ln x}{x^n}, n\in\mathbb{N} \qquad (\text{type } \frac{\infty}{\infty})$$
$$=\lim_{x\to+\infty}\frac{\frac{1}{x}}{nx^{n-1}}$$
$$=\lim_{x\to+\infty}\frac{1}{nx^n}$$
$$=0.$$

previously, when $n<m$

$\lim_{x\to+\infty}\frac{x^n}{x^m}=0$

$=\lim_{x\to+\infty}x^{n-m}$

"as $x\to+\infty$, x^n grows slower than x^m"

Remark. $\ln x\to+\infty$ slower than any x^a, $a>0$

1. L'Hôpital's rule can NOT be applied for determinate form.
   For example, $\lim_{x\to 1}\frac{x+1}{x+2}=\frac{2}{3}$, but $\lim_{x\to 1}\frac{(x+1)'}{(x+2)'}=\frac{1}{1}=1$.

2. If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ is still $\frac{0}{0}$, $\frac{\infty}{\infty}$, then repeat L'Hôpital's rule.

3. L'Hôpital's rule can be used to justify the previous assertion that as $x\to\infty$, higher degree polynomials "grows faster" than lower degree polynomials; exponential functions grow faster than any polynomials; log functions grow slower than any polynomials.

comparative growth rate:
$\ln x < x^n < x^m < e^x$
(a^x)

Exercise 6.1.1. $\lim_{x\to1}(x-1)=0=\lim_{x\to1}\ln x$

1. $\lim_{x\to 1}\frac{x-1}{\ln x}=1$ type "$\frac{0}{0}$" by L'Hôpital $=\lim_{x\to1}\frac{1}{\frac{1}{x}}=\lim_{x\to1}x=1$

2. $\lim_{x\to+\infty}\frac{x^n}{e^x}=0$

when $n<0$ $\lim_{x\to+\infty}\frac{x^n}{e^x}=\lim_{x\to+\infty}\frac{0}{e^x}=0$ when $n=0$ $\lim_{x\to+\infty}\frac{1}{e^x}=0$ when $n>0$

$\lim_{n\to\infty}x^n=+\infty=\lim_{n\to\infty}e^x$

apply L'Hôpital's rule

$=\lim_{x\to+\infty}\frac{nx^{n-1}}{e^x}=\begin{cases}0 & \text{if } n\le 1\\ \text{otherwise} & \end{cases}$

$=0$ by L'Hôpital repeated.

apply L'Hôpital rule's again

Example 6.1.3. (Applying L'Hôpital's rule twice.)

$\lim_{x\to0}(e^x-e^{-x}-2x)=0$

$\lim_{x\to0}x^2=0$

$\frac{d}{dx}e^{-x}=-e^{-x}$

$\lim_{x\to0}(e^x+e^{-x}-2)=0$

$\lim_{x\to0}(2x)=0$

$$\begin{aligned}
&\lim_{x\to0}\frac{e^x-e^{-x}-2x}{x^2} && (\text{type } \tfrac{0}{0})\\
=&\lim_{x\to0}\frac{e^x+e^{-x}-2}{2x} && (\text{ still of type } \tfrac{0}{0})\\
=&\lim_{x\to0}\frac{e^x-e^{-x}}{2}\\
=&0
\end{aligned}$$

6.1.2 Other Indeterminate Forms: $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0

All these forms can be converted to forms of types $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 6.1.4. Type $0\cdot\infty$

$$\begin{aligned}
&\lim_{x\to0^+}(x\ln x) && (0\cdot\infty)\\
=&\lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}} && (\tfrac{\infty}{\infty})\\
=&\lim_{x\to0^+}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\frac{1}{\frac{-1}{x}}\right)\\
=&\lim_{x\to0^+}(-x)\\
=&0.
\end{aligned}$$

$\lim_{x\to0^+}x=0$ $\lim_{x\to0^+}\ln x=-\infty$

$\lim_{x\to0^+}\frac{1}{x}=+\infty$

Example 6.1.5. Type $\infty - \infty$

$\lim_{x\to 0^+} \frac{1}{x} = +\infty$ $\lim_{x\to 0^+} \frac{1}{e^x-1} = +\infty$

$\frac{(e^x-1)}{(e^x-1)}\frac{1}{x} - \frac{1}{(e^x-1)}\frac{x}{x}$

$\lim_{x\to 0^+}(e^x-1-x) = 0$

$\lim_{x\to 0^+} x(e^x-1) = 0$

$\lim_{x\to 0^+}(e^x-1) = 0$

$\lim_{x\to 0^+}(e^x-1+xe^x) = 0$

$$
\begin{aligned}
&\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x-1}\right) \qquad (\infty - \infty)\\
=&\lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)} \qquad \left(\frac{0}{0}\right)\\
=&\lim_{x\to 0^+}\frac{e^x-1}{e^x-1+xe^x} \qquad \left(\text{ still } \frac{0}{0}\right)\\
=&\lim_{x\to 0^+}\frac{e^x}{e^x+e^x+xe^x}\\
=&\frac{1}{2}.
\end{aligned}
$$

Example 6.1.6. Types 1^∞, ∞^0, 0^0

Trick: $\boxed{f^g = e^{\ln f^g} = e^{g\ln f}}$

1.

$x = e^{\ln x}$

$x^{\frac{1}{x}} = (e^{\ln x})^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$

$$
\begin{aligned}
&\lim_{x\to+\infty} x^{\frac{1}{x}} \qquad (\infty^0)\\
=&\lim_{x\to+\infty} e^{\ln(x^{\frac{1}{x}})}\\
=&\lim_{x\to+\infty} e^{\frac{1}{x}\ln x}\\
=&e^{\lim_{x\to+\infty}\frac{1}{x}\ln x},
\end{aligned}
$$

use the fact that e^y is continuous $(y = \frac{1}{x}\ln x)$ $c = \lim_{x\to+\infty} y$

$\lim_{y\to c} e^y = e^c$

$$
\begin{aligned}
&\lim_{x\to+\infty}\frac{1}{x}\ln x \qquad (0\cdot\infty)\\
=&\lim_{x\to+\infty}\frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty}\right)\\
=&\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}\\
=&0.
\end{aligned}
$$

So,

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = e^0 = 1.$$

2.

$x = e^{\ln x}$

$x^{\frac{1}{1-x}} = e^{\frac{\ln x}{1-x}}$

$$
\begin{aligned}
&\lim_{x\to 1^+} x^{\frac{1}{1-x}} \qquad (1^\infty)\\
=&\lim_{x\to 1^+} e^{\frac{1}{1-x}\ln x}\\
=&e^{\lim_{x\to 1^+}\frac{\ln x}{1-x}},
\end{aligned}
$$

$$\begin{aligned} &\lim_{x\to1^+}\frac{\ln x}{1-x} \qquad (\frac{0}{0}) \\ =&\lim_{x\to1^+}\frac{\frac{1}{x}}{-1} \\ =&-1. \end{aligned}$$

So,

$$\lim_{x\to1^+} x^{\frac{1}{1-x}} = e^{-1}.$$

3.

$$\begin{aligned} &\lim_{x\to0^+} x^x \qquad (0^0) \\ =&\lim_{x\to0^+} e^{x\ln x} \\ =&e^{\lim_{x\to0^+} x\ln x}, \end{aligned}$$

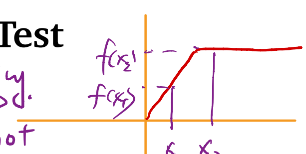
$$\begin{aligned} &\lim_{x\to0^+} x\ln x \qquad (0\cdot\infty) \\ =&\lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}} \qquad (\frac{\infty}{\infty}) \\ =&\lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ =&\lim_{x\to0^+}(-x) \\ =&0. \end{aligned}$$

So,

$$\lim_{x\to0^+} x^x = e^0 = 1.$$

6.2 Monotonicity of Functions and the First Derivative Test

increasing.

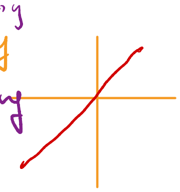


6.2.1 Monotonicity: Increasing/Decreasing Functions

Eg, but not strictly increasing

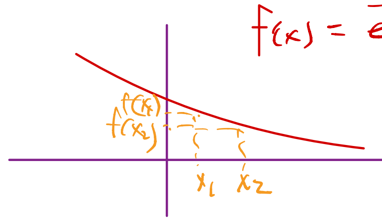
Definition 6.2.1. Let $f(x)$ be a function defined on (a, b). Then

E.g. f(x)=x is increasing also strictly increasing



1. $f(x)$ is **increasing** (or *positively monotone*) on the interval if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$.
2. $f(x)$ is **strictly increasing** (or *strictly positive monotone*) on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

$f(x) = e^{-x}$



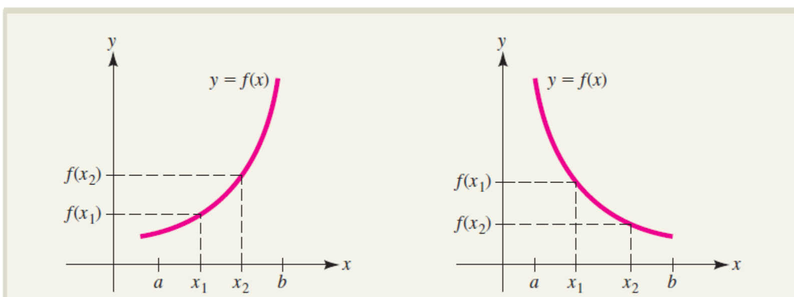
decreasing and strictly decreasing

3. $f(x)$ is **decreasing** (or *negatively monotone*) on the interval if $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$.

4. $f(x)$ is **strictly decreasing** (or *strictly negative monotone*) on the interval if $f(x_2) < f(x)$ whenever $x_2 > x_1$.

5. $f(x)$ is *(strictly) monotone* if $f(x)$ is either (strictly) positively monotone or (strictly) negatively monotone.

Caveat! The preceding definition is the mathematicians' definition of increasing/decreasing functions. However, some calculus texts define increasing/decreasing functions differently, e.g. [Hoffmann et al.], where "increasing/descreasing functions" refer to the "strictly increasing/descreasing functions" defined above. Similarly, some text refers to what we called "strictly monotone/monotone" above as "monotone/weakly monotone".



(a) $f(x)$ is increasing on $a < x < b$ (b) $f(x)$ is decreasing on $a < x < b$

$f' \geq 0.$
f is increasing

Theorem 6.2.1. *Let f be a differentiable function on (a, b).*

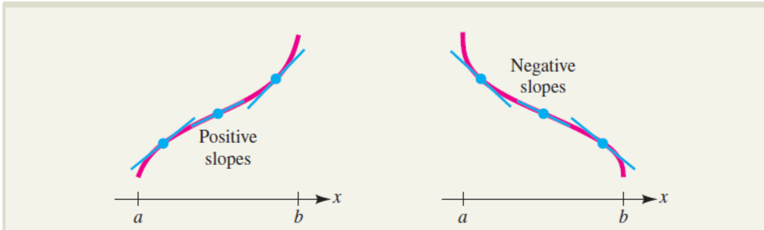
1. *If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f(x)$ is an increasing function.*
2. *If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly increasing function on (a, b).*
3. *If $f'(x) \leq 0$ for all $x \in (a, b)$, then $f(x)$ is a decreasing function.*
4. *If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is a strictly decreasing function on (a, b).*

Example 6.2.1. Show that $f(x) = e^x - x - 1$ is a strictly increasing function on $(0, \infty)$.

Solution. $f'(x) = e^x - 1 > 1 - 1 = 0$. So $f(x)$ is a strictly increasing function. ■

Remark. Because $f(x)$ is a strictly increasing function, $f(x) > f(0) = 0$ for $x > 0$, i.e.

$$e^x > 1 + x, \quad \text{for } x > 0.$$



(a) $f'(x) > 0$ on $a < x < b$, so $f(x)$ is increasing.
(b) $f'(x) < 0$ on $a < x < b$, so $f(x)$ is decreasing.

Procedure to determine intervals of increase/decrease of f

i.e. critical points of f.

1. Find all c such that $f'(c) = 0$ or $f'(c)$ is undefined. Divide the line into several intervals.
2. For each intervals (a, b) obtained in the previous step.
   (a) If $f'(x) > 0$, $f(x)$ is a strictly increasing function (↑) on (a, b).
   (b) If $f'(x) < 0$, $f(x)$ is a strictly decreasing function (↓) on (a, b).

Example 6.2.2. Find the intervals in which the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7$$

is strictly increasing/strictly decreasing.

f'	+	−	+
$x-1$	−	−	+
$x+2$	−	+	+
	-2		1

Solution.

$$f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1) = 0 \quad \Rightarrow x = -2, 1.$$

So we have 3 intervals: $(-\infty, -2)$, $(-2, 1)$, $(1, \infty)$.

$$\begin{array}{lll} \text{In } (-\infty, -1), & x+1 < 0, x-1 < 0, & \text{so } f'(x) > 0. \\ \text{In } (-1, 1), & x+1 > 0, x-1 < 0, & \text{so } f'(x) < 0. \\ \text{In } (1, +\infty), & x+1 > 0, x-1 > 0, & \text{so } f'(x) > 0. \end{array}$$

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
monotonicity	↑		↓		↑

■

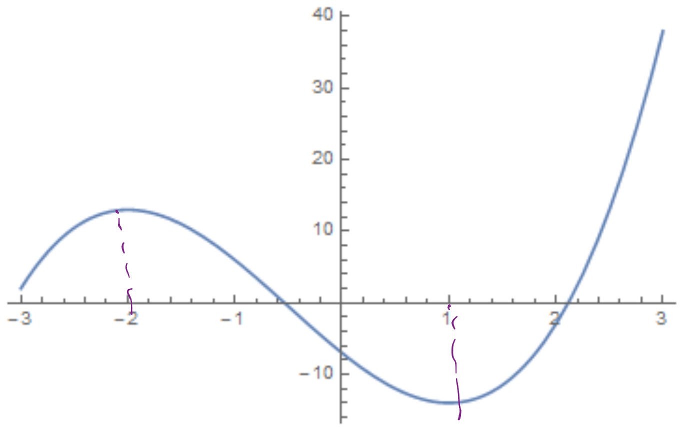


Figure 6.1: $y = 2x^3 + 3x^2 - 12x - 7$

Exercise 6.2.1. Find the intervals of strict increase and strict decrease of the function

$$f(x) = x^7 - 2x^5 + x^3.$$

$7x^2-3 \gtrless 0$ when $|x| \gtrless \sqrt{\frac{3}{7}}$

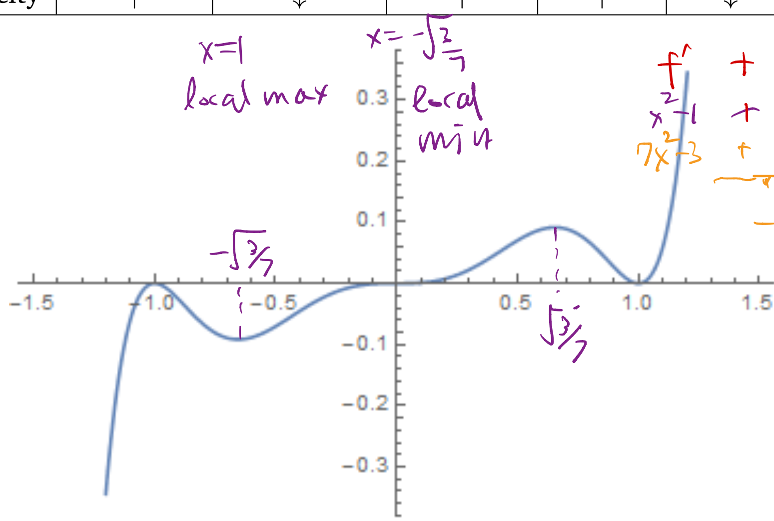
Solution.

$= x^2(7x^4-10x^2+3)$ $x^2-1 \gtrless 0$ when $|x| \gtrless 1$

$$f'(x) = 7x^6-10x^4+3x^2 = x^2(7x^2-3)(x^2-1) = 0 \quad \Rightarrow \quad x = 0, \pm 1 \text{ and } \pm\sqrt{\frac{3}{7}} \approx \pm 0.654654.$$

>0 when $x \neq 0$ $x=\sqrt{\frac{3}{7}}$ local max $x=1$ local min

x	$(-\infty,-1)$	$(-1,-\sqrt{\frac{3}{7}})$	$(-\sqrt{\frac{3}{7}},0)$	$(0,\sqrt{\frac{3}{7}})$	$(\sqrt{\frac{3}{7}},1)$	$(1,+\infty)$
$f'(x)$	$+$	$-$	$+$	$+$	$-$	$+$
monotonicity	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$



f'	$+$	$-$	$+$	$+$	$-$	$+$
x^2-1	$+$	$-$	$-$	$-$	$-$	$+$
$7x^2-3$	$+$	$+$	$-$	$-$	$+$	$+$
	-1	$-\sqrt{3/7}$	0	$\sqrt{3/7}$	1	

Figure 6.2: $y = x^7 - 2x^5 + x^3$

■

Definition 6.2.2. Let $f(x)$ be a real-valued function defined on (a,b). A number $c \in (a,b)$ is called a critical point of f if $f'(c) = 0$ or $f'(c)$ does not exist.
The corresponding value $f(c)$ is called a critical value for $f(x)$.

Remark. The notion of critical points applies to more general functions, e.g. real functions of several variables, complex functions etc. A critical point always lies in the domain of the function. In the special case of real-valued functions of a single real variable, a critical point is a real number; therefore it is also called a *critical number*. Let $f(x)$ be a real-valued function of a single real variable, and $c \in \mathbb{R}$ be a critical point of f. Let $C \subset \mathbb{R}^2$ be the graph of f in the $x-y$ plane. The point $(c, f(c)) \in C$ is a critical point of the function $\pi_y : C \to \mathbb{R}$ given by $(x,y) \mapsto y$.

Example 6.2.3.

$$f(x) = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

$$f(x) = |x|.$$

We have proved

$$f'(x) = \begin{cases} -1, & x < 0, \\ \text{does not exist}, & x = 0, \\ 1, & x > 0. \end{cases}$$

$\Rightarrow$ critical number: $x = 0$;   corresponding critical value: 0

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	$\cancel{0}$ does not exist	$+$
monotonicity of f.	$\downarrow$		$\uparrow$

Example 6.2.4. $f(x) = x^4 - 4x^3$. Find all critical points and increasing & decreasing intervals.

Solution.

$$f'(x) = 4x^3 - 12x^2 = \underbrace{4x^2}_{>0 \text{ when } x \neq 0}(x-3) = 0 \quad \Rightarrow \quad x = 0, 3.$$

critical points: $x = 0, 3$
corresponding critical values: $f(0) = 0,\ f(3) = -27$

x	$(-\infty, 0)$	0	$(0,3)$	3	$(3, +\infty)$
$f'(x)$	$-$	0	$-$	0	$+$
monotonicity	$\downarrow$		$\downarrow$		$\uparrow$

■

6.2.2 Maxima & Minima of Functions

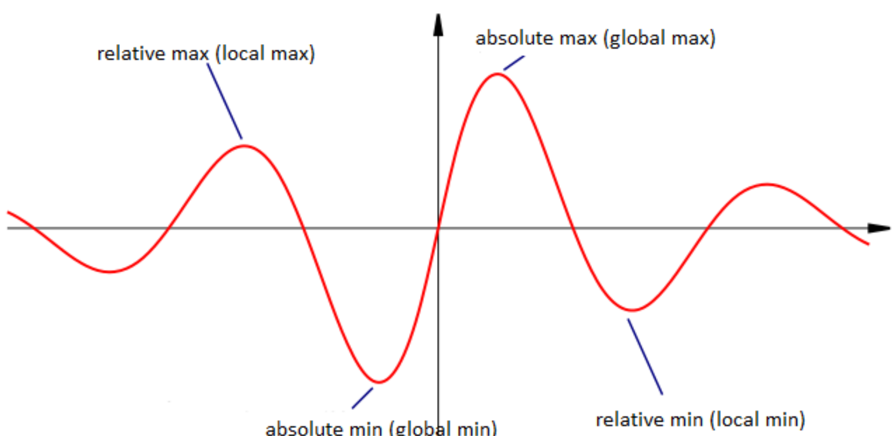
Definition 6.2.3. Let $f(x)$ be a real-valued function with domain I. We say

1. $f(x)$ has a **relative maximum** (or **local maximum**) at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$ near c.

2. $f(x)$ has a **global maximum** (or **absolute maximum**) at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$.

Similar definition for relative/global minimum.

Both maximum and minimum are called an extremum.



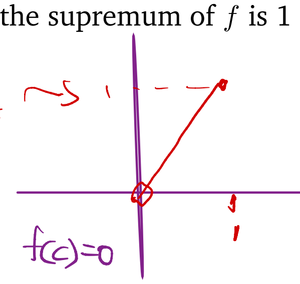
and local min. but not abs min.

Remark. Global extremum $\Rightarrow$ Local extremum
But Global extremum $\not\Leftarrow$ Local extremum

Remark. There is some confusion in the literature regarding whether a (local or global) maximum/minimum of a function refers to an element in the domain or its corresponding value (in the range). For most literature, *the* (absolute) maximum of a real function $f(x)$ refers to the value: *$M \in \mathbb{R}$ is said to be the (absolute) maximum if there exists an element c in the domain D of f such that $f(x) \leq f(c)$ $\forall x \in D$.* To be clear, say that M is an (absolute) maximum *value* of f; and f *attains* its (absolute) maximum *at* c. Say e.g. f has local maxima at $x_1, x_2, \ldots \in D$, with corresponding values $f(x_1), f(x_2), \ldots$. Similarly for the notions of (absolute/local) minimum.

Remark. Absolute maxima/minima may not exist. Consider the e.g. the function $f : (0, 1] \to \mathbb{R}$ given by $f(x) = x$. This f has an absolute maximum but has no absolute minimum. A general notion is *supremum/infimum*. In the above example, the supremum of f is 1 and its infimum is 0.

abs max $\rightsquigarrow$ 1



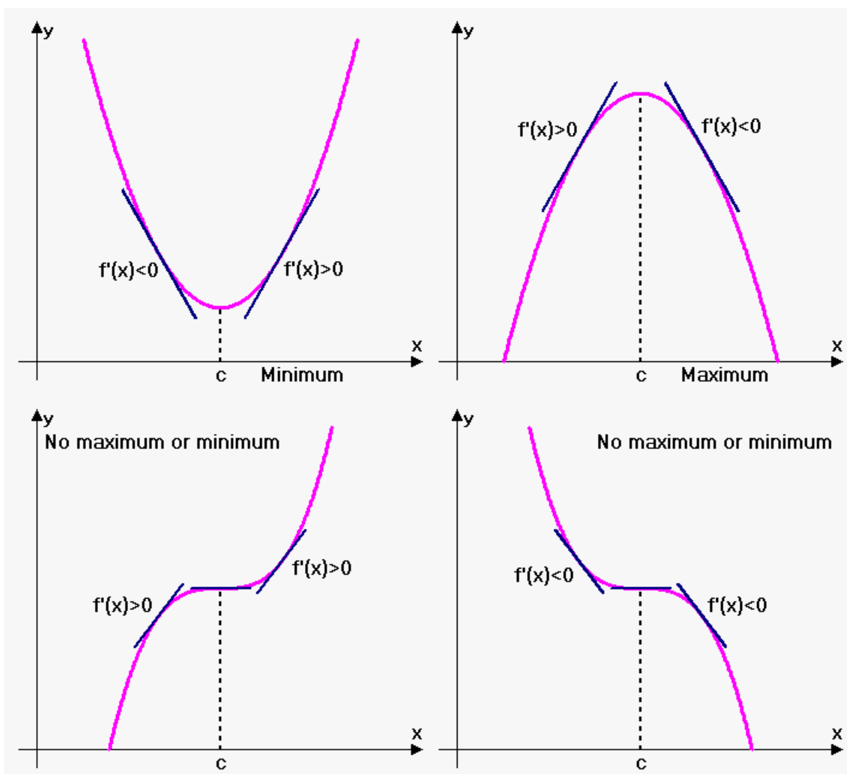
no absolute minimum
(f(x) can be arbitrarily close to 0
but there is no c such that f(c)=0

Question I: How to find relative extrema?

Theorem 6.2.2 (First Derivative Test: Relative Extrema)**.**

Let $f(x)$ be a continuous function which is differentiable where $x \neq c$. Then

1. *$f(x)$ attains a relative maximum at $x = c$ if near the point c,*
$$f'(x) > 0 \quad \text{for } x < c; \quad f'(x) < 0 \quad \text{for } x > c.$$
2. *$f(x)$ attains a relative minimum at $x = c$ if near the point c,*
$$f'(x) < 0 \quad \text{for } x < c; \quad f'(x) > 0 \quad \text{for } x > c.$$
3. *$f(x)$ attains no relative extremum at $x = c$ if near the point c, $f'(x)$ has the same sign on two sides of c.*



Property	Sign of $f'(x)$ to the left of c	Sign of $f'(x)$ to the right of c
Relative maximum	$+$	$-$
Relative minimum	$-$	$+$
Not a relative extremum	$+$	$+$
Not a relative extremum	$-$	$-$

Theorem 6.2.3. *Let $c \in (a, b)$ and let f be a continuous function on (a, b) such that f' exists and is continuous on $(a, b)\backslash\{c\}$. Then f attains a relative extremum at $x = c$ $\Rightarrow$ c is a critical number, i.e. $f'(c) = 0$ or $f'(c)$ does not exist.*

Remark. f attains a relative extremum at $x = c$ $\not\Leftarrow$ c is a critical number.
For example, $f(x) = x^3$, $f'(x) = 3x^2$, so $x = 0$ is a critical number. But $f'(x) > 0$ on two sides of $x = 0$, so f does not have a relative extremum at 0.

Example 6.2.5. Let

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

Find all its relative maxima and relative minima.

Solution. Refer to the answer of Example 6.2.2, $f'(x) = 6x^2 + 6x - 12$. The critical numbers are solutions of $f'(x) = 0$, i.e $x = -2$ and $x = 1$.

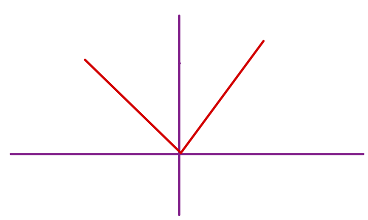
x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$

↑local max ↑local min

(point where a relative maximum occurs, corresponding value): $(-2, f(-2)) = (-2, 13)$
(point where a relative minimum occurs, corresponding value): $(1, f(1)) = (1, 14)$

■

Example 6.2.6.



$$f'(x) = \begin{cases} - & \text{when } x<0 \\ + & \text{when } x>0 \end{cases}$$

1. For Example 6.2.3 $f(x) = |x|$.
   One critical number: $\underline{x = 0}$, One relative minimum at 0, with corresponding value 0.
2. For example 6.2.4 $f(x) = x^4 - 4x^3$.
   critical numbers: $x = 0, 3$, one relative minimum at 3, with corresponding value -27.

Exercise 6.2.2. Let

$$f(x) = x^7 - 2x^5 + x^3.$$

(see Exercise 6.2.1) Find all its relative maxima and relative minima of f.

Answer:

$$\begin{aligned}
&\text{(point where a relative maximum occurs, corresponding value)}: \\
&\qquad (-1, f(-1)) = (-1, 0); (\sqrt{\tfrac{3}{7}}, f(\sqrt{\tfrac{3}{7}}) \approx (0.655, 0.092) \\
&\text{(point where a relative minimum occurs, corresponding value)}: \\
&\qquad (-\sqrt{\tfrac{3}{7}}, f(-\sqrt{\tfrac{3}{7}})) \approx (-0.655, -0.092); (1, f(1)) = (1, 0).
\end{aligned}$$

Note that f has no relative extremum at 0.

Question II: How to find absolute Max/Min?

Theorem 6.2.4. *Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function, then the absolute maximum point and absolute minimum point exist for the graph of f (Theorem 3.2.2 Extreme Value Theorem).*

Remark. Note that the preceding theorem applies only when the domain of f is a *closed finite* interval!

Procedures to find absolute max/min of continuous function f on $[a, b]$

1. Find all the critical numbers $c_1, c_2, \dots,$ in (a, b).
2. Compute the values $f(a), f(b), f(c_1), f(c_2), \dots,$
   The maximum value corresponds to the absolute max.
   The minimum value corresponds to the absolute min.

Example 6.2.7. Find the absolute maximum and absolute minimum of $f(x) = x^5 - 80x$ on $[-3, 4]$.

Solution. Since $f(x)$ is continuous on $[-3, 4]$, the absolute max/min can be reached by extreme value theorem.

$$\text{''} \; 5(x^4-16) = 5(x^2+4)(x^2-4) = 5(x^2+4)(x+2)(x-2)$$

$$f'(x) = 5x^4 - 80 = 0 \quad \Rightarrow \quad x = -2, 2.$$

Compute

$$f(-2) = 128, \quad f(2) = -128, \leftarrow \text{abs. min.}$$
$$f(-3) = -3, \quad f(4) = 704. \leftarrow \text{absolute maxi}$$

The absolute minimum is -128, attained at $x = 2$; the absolute maximum is 704, attained at $x = 4$.

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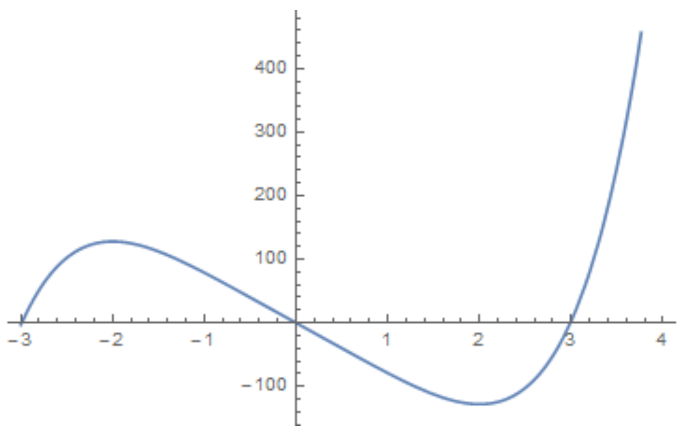


Figure 6.3: $y = x^5 - 80x$ over $[-3, 4]$

E.g., $f(x) = |x|$ defined on $[-1, 2]$

one critical point: $x = 0$

$f(0) = 0 \leftarrow$ abs min. attained at $x = 0$

$f(-1) = 1$

$f(2) = 2. \leftarrow$ abs max, attained at $x = 2$

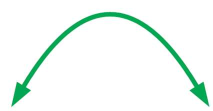
MATH1520 University Mathematics for Applications **Fall 2021**

Chapter 7: Application of Derivatives II

Learning Objectives:
(1) Discuss concavity.
(2) Use the sign of the second derivative to find intervals of concavity.
(3) Locate and examine inflection points.
(4) Apply the second derivative test for relative extrema.
(5) Determine horizontal and vertical asymptotes of a graph.
(6) Discuss and apply a general procedure for sketching graphs.

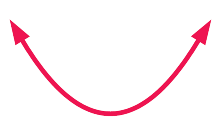
7.1 Concavity and points of inflection

Intuitively: On the $x - y$ plane: when a curve, or part of a curve, has the shape:



concave

we say that the shape is concave downward. On the other hand, if it takes the shape



convex

we say that it is concave upward.

Remark. In some textbooks "concave upward" is called concave up or convex; "concave downward" is called concave down or concave.

Definition 7.1.1. If the function $f(x)$ is differentiable on the interval (a, b), then *the graph* of f is

(strictly convex)
(i) strictly concave upward on (a,b) if $f'(x)$ is strictly increasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) > 0$.

(strictly concave)
(ii) strictly concave downward on (a,b) if $f'(x)$ is strictly decreasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) < 0$.

(convex)
(iii) concave upward on (a,b) if $f'(x)$ is increasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) \geq 0$.

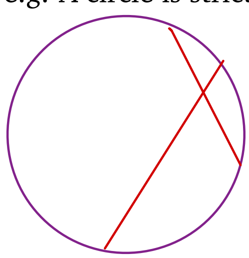
(concave)
(iv) concave downward on (a,b) if $f'(x)$ is decreasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) \leq 0$.

In case (i)/(iii), the function f is said to be *strictly convex/convex*; in case (ii)/(iv), f is said to be *strictly concave/concave*.

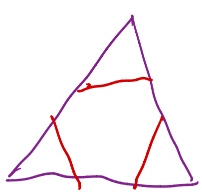
Remark. 1. In some calculus texts, what we called "strictly convex/concave" above is called "convex/concave", and what we called " convex/concave" above is called "weakly convex/concave"

2. General definition of convexity/concavity of continuous curves on a plane via secant lines:

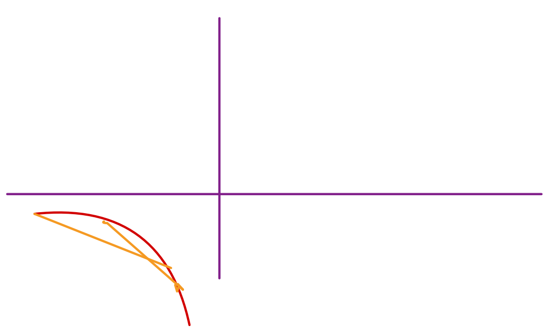
• *For a closed curve $C \subset \mathbb{R}^2$: C* is strictly convex if all secant lines to C lies in the "inside" except for the end points. e.g. A circle is strictly convex.



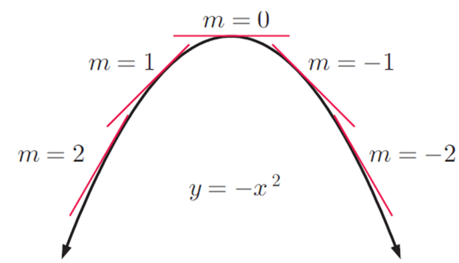
E.g. A piecewise convex curve:



• *For the graph C of a continuous function f on the $x-y$ plane:* f is concave if all secant lines to the graph do not intercept the "upside component" of $\mathbb{R}^2\backslash C$. E.g. $C = \{(x,y)\,|\,f(x) = \dfrac{1}{x},\, x < 0\}$.



A test for shapes of graphs:



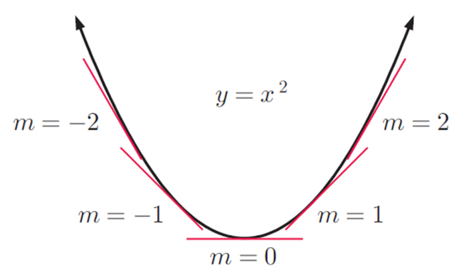
concave

$f' = -2x$ strictly decreasing.

$f'' = -2$

As x increases, $f'(x)$ is $\downarrow$

$f''(x) = -2 < 0$ for strictly concave downward curve.



convex

$f' = 2x$ strict increasing

$f'' = 2$

As x increases, $f'(x)$ is $\uparrow$

$f''(x) = 2 > 0$ for strictly concave upward curve.

Definition 7.1.2. If $f(x)$ changes strict concavity at some point c in the domain, then the point $(c, f(c))$ on the $x - y$ plane is called an *inflection point* of the graph of f.

Procedure for Determining Intervals of Concavity & Inflection Points:

Suppose the function $f(x)$ is such that f'' is piecewise continuous.

1. Find all c for which $f''(c) = 0$ or $f''(c)$ does not exist, and divides the domain into several intervals.
2. For each interval,
   - if $f''(x) > 0$, the graph of $f(x)$ is strictly concave upward. (I.e. f is a convex function.)
   - if $f''(x) < 0$, the graph of $f(x)$ is strictly concave downward. (I.e. f is a concave function.)
3. For all c found in step 1,
   - if $f''(x)$ changes sign on two sides of c, then $(c, f(c))$ is an inflection point on the graph of f;
   - otherwise, $(c, f(c))$ is not an inflection point on the graph of f.

Example 7.1.1.

$$f(x) = x^3 + 1 \qquad f' = 3x^2$$
$$f''(x) = 6x = 0 \quad \Rightarrow \quad x = 0.$$

inflection pt occurs

f'' $-$ $+$

0

- if $x < 0$, $f''(x) < 0$, $\Rightarrow$ f is strictly concave on $(-\infty, 0)$;
- if $x > 0$, $f''(x) > 0$, $\Rightarrow$ f is strictly convex on $(0, \infty)$.

$f(0) = 0+1 = 1$

Since $f''(x)$ changes signs on both sides of $x = 0$, $(0, 1)$ is the unique inflection point on the graph of f.

Example 7.1.2. Describe the concavity and find all inflection points of the graph of $f(x) = 2x^6 - 5x^4 + 7x - 3$.

Solution. $f' = 12x^5 - 20x^3 + 7$

$$f''(x) = 60x^4 - 60x^2 = 60x^2(x^2 - 1) = 60x^2(x-1)(x+1) = 0 \quad \Rightarrow \quad x = 0, \pm 1.$$

>0 when $x \neq 0$

x	$(-\infty, 0)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
$f''(x)$	$+$	0	$-$	0	$-$	0	$+$
concavity	up($\smile$)		down($\frown$)		down($\frown$)		up($\smile$)

Two inflection points: $(-1, -13)$, $\cancel{(1, 1)}$. inflection pts occur.

$((0, -3)$ is not an inflection point!)

$f(-1) = 2-5-7-3$
$= -13$

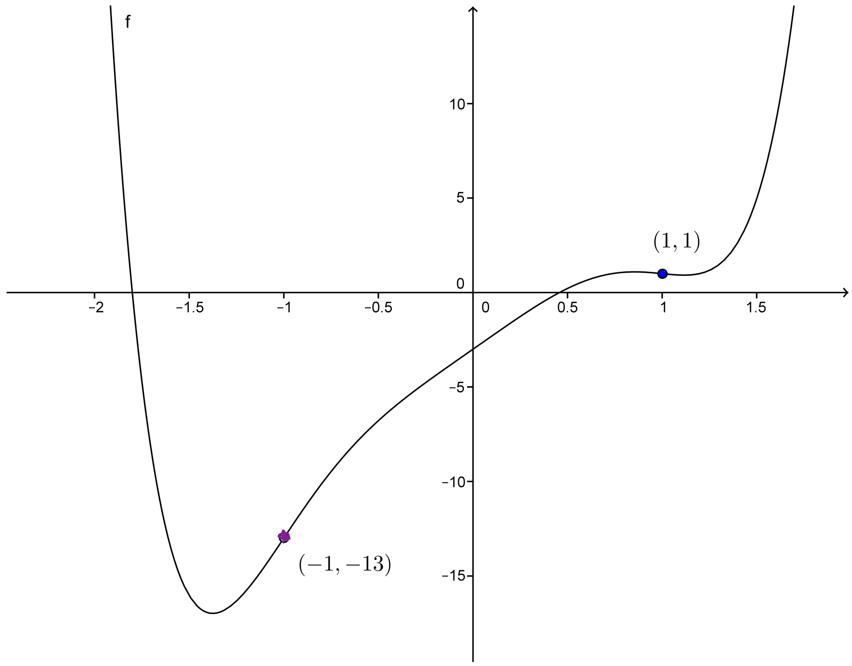
$x+1$ $-$ $+$ $+$ $+$

$x-1$ $-$ $-$ $-$ $+$

-1 0 1

$f(1) = 2-5+7-3$
$= 1$

f'' $+$ $-$ $-$ $+$



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Remark.

- c is a critical point $\iff$ $f'(c)=0$ or $f'(c)$ does not exist

  c is a critical point $\left\{\begin{array}{c} \Longleftarrow \\ \not\Longrightarrow \end{array}\right\}$ f' changes sign at c

  E.g, $f(x)=x^3$ $x=0$ is a critical pt. $f'=3x^2$ doesn't change sign at 0

- $(c,f(c))$ is an inflection point on the graph of f $\iff$ f'' changes sign at c

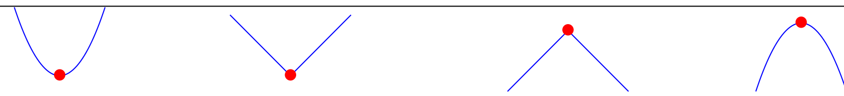
  $(c,f(c))$ is an inflection point on the graph of f $\left\{\begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array}\right\}$ $f''(c)=0$ or undefined

Theorem 7.1.1 (The Second Derivative Test: Relative Extrema).

Suppose $f'(a)=0$!

if f' changes from $+$ to $-$ $\rightarrow$ local max
" $-$ $+$ $\rightarrow$ local min.

1. *If $f''(a)<0$, then f has a relative maximum at a.*
2. *If $f''(a)>0$, then f has a relative minimum at a.*
3. *If $f''(x)=0$, we have no conclusion.*



	relative min	relative min	relative max	relative max
	$f'(a)=0$	$f'(a)$ does not exist	$f'(a)$ does not exist	$f'(a)=0$
1st test:	$-\quad+$	$-\quad+$	$+\quad-$	$+\quad-$
2nd test:	$f''(a)>0$	Not Applicable	Not Applicable	$f''(a)<0$

Example 7.1.3.

$$f(x)=\frac{1}{30}x^6-\frac{1}{12}x^4.$$

Use the first and second derivative test to study the relative extrema.

$x-\sqrt{5/3}$	$-$	$-$	$-$	$+$
$x+\sqrt{5/3}$	$-$	$+$	$+$	$+$
x^2	$-$	$-$	$+$	$+$
f'	$-$ $-\sqrt{5/3}$	$+$ 0	$-$ $\sqrt{\frac{5}{3}}$	$+$

Solution.

$$= \frac{1}{5}x^3\left(x^2-\frac{5}{3}\right)$$

$$f'(x)=\frac{1}{5}x^5-\frac{1}{3}x^3=\frac{1}{5}x^3\left(x+\sqrt{\frac{5}{3}}\right)\left(x-\sqrt{\frac{5}{3}}\right)=0 \quad\Rightarrow\quad x=-\sqrt{\frac{5}{3}},0,\sqrt{\frac{5}{3}}$$

$$f''(x)=x^2(x+1)(x-1).$$

$$= x^4-x^2 = x^2(x^2-1)$$

x	$(-\infty,-\sqrt{\frac{5}{3}})$	$-\sqrt{\frac{5}{3}}$	$(-\sqrt{\frac{5}{3}},0)$	0	$(0,\sqrt{\frac{5}{3}})$	$\sqrt{\frac{5}{3}}$	$(\sqrt{\frac{5}{3}},+\infty)$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f''(x)$		$f''>0$ $\frac{5}{3}(\frac{5}{3}-1)$		$f''=0$ $0\cdot(-1)$		$f''>0$ $\frac{5}{3}(\frac{5}{3}-1)$	
1st test:		relative min		relative max		relative min	
2nd test:		relative min		inconclusive		relative min	

■

Exercise 7.1.1. Apply the first and the second derivative tests to find the local maxima/minima and the global maximum/minimum of $f(x)=x^3-3x$.

$f'=3x^2-3=3(x+1)(x-1)$ critical pts: $x=-1,\ 1$

$x+1$	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
f'	$+$	$-$	$+$

-1: local max; 1: local min.

to find global max/min:

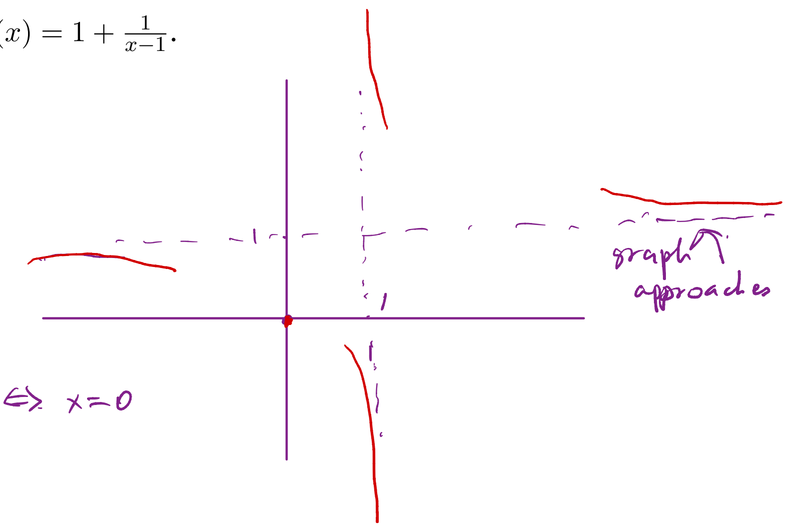
$f(1)$, $f(-1)$ finite, $\lim_{x\to\infty} f(x)=+\infty$, $\lim_{x\to-\infty} f(x)=-\infty$

$f''=6x$ $\quad f''(-1)=-6<0.$ $\quad f''(1)=6>0$

so global max/min don't exist.

7.2 Curve sketching

Example 7.2.1. Sketch the graph of $y = f(x) = 1 + \frac{1}{x-1}$.



Solution.

Step 1. Analyze $f(x)$.

1. domain: $\{x \in \mathbb{R} \,|\, x \neq 1\}$
2. x, y intercepts: $f(0) = 1 + (-1) = 0$
   Let $x = 0$, then $y = 0$;
   Let $y = 0$, then $x = 0$. $1 + \frac{1}{x-1} = 0 \Leftrightarrow x = 0$
   $\Rightarrow$ only one intercept: $(0, 0)$
3. vertical and horizontal asymptotes:

$$\lim_{x\to 1^+} f(x) = +\infty, \lim_{x \to 1^-} f(x) = -\infty \quad \Rightarrow \quad \text{vertical asymptote: } x = 1$$
$$\lim_{x\to +\infty} f(x) = 1, \lim_{x \to -\infty} f(x) = 1 \quad \Rightarrow \quad \text{horizontal asymptote: } y = 1.$$

Step 2. Analyze $f'(x)$.

$$f'(x) = -\frac{1}{(x-1)^2}, x \neq 1.$$

1. interval where f is strictly increasing: none $\quad$ ($f'(x) < 0$ in the domain)
   interval where f is strictly decreasing: $(-\infty, 1)$, $(1, +\infty)$
2. critical points of f: none $\quad$ ($x = 1$ is not in the domain)
3. relative extrema of f: none

Step 3. Analyze $f''(x)$.

$$f''(x) = \frac{2}{(x-1)^3}, x \neq 1.$$

1. interval where f is strictly convex: $(1, +\infty)$ $\quad$ ($f'' > 0$)
   interval where f is strictly concave: $(-\infty, 1)$ $\quad$ ($f'' < 0$)
2. inflection points on the graph: none $\quad$ ($x = 1$ is not in the domain)